Notes.

I.

On Symbols of Operation.

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To prove that, ϕ being any function of D, *i. e.* $\dfrac{d}{dx}$,

$$e^{x\phi(D)}e^{hx} = e^{\lambda x}, \tag{1}$$

where λ is a constant determined from

$$\psi(\lambda) = 1 + \psi(h), \tag{2}$$

the function ψ being defined by

$$\psi(x) = \int \frac{dx}{\varphi(x)}. \tag{3}$$

The above theorem might be made to follow from principles given in a paper by me in the Proceedings of the London Mathematical Society, April 1881; but the following method may be also employed.

Let $$u = e^{kx\phi(D)}e^{hx};$$

differentiating with regard to h,

$$\frac{du}{dh} = e^{kx\phi D}xe^{hx},$$

also

$$\frac{du}{dk} = e^{kx\phi D}x\phi(D)e^{hx}$$
$$= \phi(h)e^{kx\phi D}xe^{hx}.$$

We have thus a partial differential equation

$$\frac{du}{dk} = \phi(h)\frac{du}{dh}$$

$$\therefore \quad u = \chi\left(k + \int\frac{dh}{\varphi(h)}\right) = \chi\left(k + \psi(h)\right).$$

To determine the arbitrary function χ, we observe that if $k = 0$, $u = e^{hx}$;

$$\therefore \quad \chi(\psi h) \equiv e^{hx}$$

hence if λ be determined so as to satisfy

$$\psi(\lambda) \equiv k + \psi(h)$$
$$u = \chi(\psi\lambda) = e^{\lambda x}.$$

For instance, let $\phi(D) = D^2$; $\psi(x) = -\frac{1}{x}$ by (3); hence by (2), $\lambda = \frac{h}{1-h}$;

$$\therefore \quad e^{xD^2} e^{hx} = e^{\frac{hx}{1-h}}$$

Again $$e^{kxD^{-1}} e^{hx} = e^{x\sqrt{h^2+2k}}.$$

Again $$e^{kxD^r} e^{hx} = e^{\lambda x},$$

where $$\lambda = [h^{1-r} + (1-r)k]^{\frac{1}{1-r}}.$$

The above process may sometimes be applied in other similar cases; for example, to find

$$u = e^{kx^{-1}D^2} e^{hx^3}:$$

we may deduce the equation

$$\frac{du}{dk} = 9h^2 \frac{du}{dh} + 6hu;$$

the solution of this by Lagrange's method, or otherwise, is

$$u = h^{-\frac{2}{3}} \phi(h^{-1} - 9k);$$

and determining ϕ from the condition that $u = e^{hx}$ when $k = 0$, we find

$$u = (1 - 9hk)^{-\frac{2}{3}} \exp. \left(\frac{hx^3}{1 - 9hk}\right).$$

A very remarkable identity between certain symbolic operators of the exponential form may be established by a comparison of Lagrange's theorem and a result which I have given in the paper above referred to. It is there shown that if z is related to x by the equation

$$\psi(z) = 1 + \psi(x) \quad \ldots . \quad (1)$$

and if we put for shortness

$$\frac{1}{\psi'(x)} = \phi(x) = \phi \quad \ldots . \quad (2)$$

then $$e^{\phi D} F(x) = F(z) \quad \ldots . \quad (3)$$

also $$e^{D\phi} F(x) = F(z) \frac{dz}{dx} \quad \ldots . \quad (4)$$

whatever be the function F.

Now if we use for shortness the symbol $e^{D\cdot f(x)}$ or $e^{D\cdot f}$ to express the development

$$e^{D\cdot f} \equiv 1 + Dfx + \frac{1}{1.2} D^2 (fx)^2 + \&c., \quad \ldots . \quad (5)$$

Lagrange's theorem gives

$$e^{D\cdot f} F(x) \equiv F(z) \frac{dz}{dx} \quad \ldots . \quad (6)$$

where $$z = x + f(z) \quad \ldots . \quad (7)$$

Hence if z be the same function of x in (6) as in (4), the operator $e^{D\phi}$ is identical with $e^{D \cdot f}$; that is

$$1 + D\phi + \frac{1}{1\,.\,2}\overline{D\phi}|^2 + \&c. \equiv 1 + Df + \frac{1}{1\,.\,2}D^2f^2 + \&c. \quad \ldots\ldots \quad (8)$$

In order that this identity should hold, it is necessary that z, $f(x)$, $\psi(x)$ shall be three functions of x which satisfy (1) and (7), viz.

$$z = x + f(z) \quad \ldots\ldots \quad (7)$$
$$\psi(z) = 1 + \psi(x) \quad \ldots\ldots \quad (1)$$

$\phi(x)$ being put for $\dfrac{1}{\psi'(x)}$.

If we could then eliminate (z) from (1) and (7), any functions f, ψ, whose forms satisfy the resulting equation, will cause the identity (8) to hold.

For instance, suppose $\phi(x) = hx^2$, then $\psi x = -\dfrac{1}{hx}$;

hence (1)

$$x = \frac{z}{1 + hz}$$

therefore (7)

$$f(z) \equiv \frac{hz^2}{1 + hz}$$

therefore $1 + hDx^2 + \dfrac{h^2}{1\,.\,2}\overline{Dx^2}|^2 + \&c. \equiv 1 + hD\dfrac{x^2}{1 + hx} + \dfrac{h^2}{1\,.\,2}D^2\left(\dfrac{x^2}{1 + hx}\right)^2 + \&c.$

whatever be the operand.

Again let us suppose $f(x) = hx$, then $z = \dfrac{x}{1 - h}$;

hence (7) ψ is to be found from

$$\psi\left(\frac{x}{1 - h}\right) \equiv 1 + \psi(x)$$

$$\therefore \quad \psi(x) \equiv -\frac{\log x}{\log(1 - h)}$$

$$\therefore \quad \phi(x) = -x\log(1 - h)$$

Thus

$$1 + hDx + \frac{h^2}{1.2}D^2x^2 + \ \ldots\ldots \equiv e^{-\log(1-h)Dx} \equiv (1 - h)^{-Dx}$$

$$\equiv 1 + hDx + \frac{h^2}{1.2}Dx(Dx + 1) + \frac{h^3}{1.2.3}Dx(Dx + 1)(Dx + 2) + \ \ldots\ldots$$

This is easy to verify.